The structure and labelled enumeration of $K_{3,3}$-subdivision-free projective-planar graphs*

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Abstract

We consider the class $\mathcal{F}$ of 2-connected non-planar $K_{3,3}$-subdivision-free graphs that are embeddable in the projective plane. We show that these graphs admit a unique decomposition as a graph K_5 (the *core*) where the edges are replaced by two-pole networks constructed from 2-connected planar graphs. A method to enumerate these graphs in the labelled case is described. Moreover, we enumerate the homeomorphically irreducible graphs in $\mathcal{F}$ and homeomorphically irreducible 2-connected planar graphs. Particular use is made of two-pole (directed) series-parallel networks. We also show that the number m of edges of graphs in $\mathcal{F}$ satisfies the bound $m \le 3n - 6$, for $n \ge 6$ vertices.

1 Introduction

The *projective plane* is a non-orientable surface of non-orientable genus 1 that can be represented as a circular disk with its antipodal points identified. A graph G is *projective-planar* if it can be drawn on the projective plane without any pair of edges crossing. See Figure 1 where two projective planar embeddings of K_5 are represented. A graph G is projective-planar if and only if it contains at most one non-planar projective-planar 2-connected component while all the other 2-connected components of G are planar.

In this paper, we consider the class $\mathcal{F}$ of 2-connected non-planar projective-planar graphs without a $K_{3,3}$-subdivision. Results for the class $C_{\mathcal{F}}$ of connected projective-planar (non-planar) graphs with no $K_{3,3}$-subdivisions are then easily deduced. Since $K_{3,3}$

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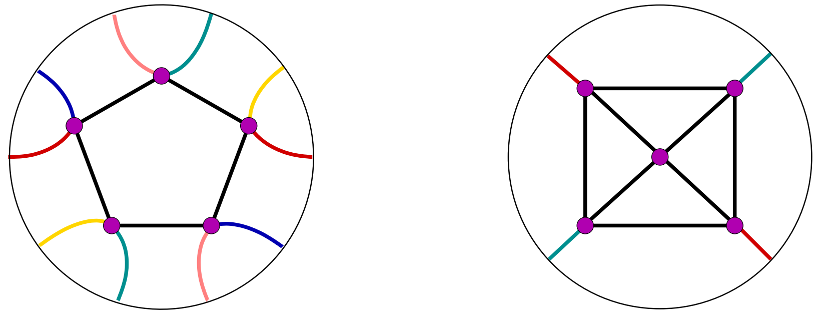

Figure 1: *Embeddings of K_5 in the projective plane.*

is a 3-regular graph, it is possible to see that the graphs with no $K_{3,3}$-subdivisions are precisely the graphs with no $K_{3,3}$-minors. Therefore we may refer to them as $K_{3,3}$-*free graphs*. By Kuratowski's Theorem [15], these graphs must contain a subdivision of K_5. Hence the simplest graphs G in $\mathcal{F}$ consist of a graph K_5 (the *core*) where the edges are replaced by two-pole networks constructed from 2-connected planar graphs. We show that in fact this is the only possibility and moreover that the core K_5 of G is uniquely determined as well as the two-pole networks entering in the construction. This fact is expressed in Theorem 1 by the equation

$$\mathcal{F} = K_5 \uparrow \mathcal{N}_P. \tag{1}$$

This property is specific to the projective plane since for other surfaces, for instance, for the torus, more complex cores such as *toroidal crowns* can occur (see [10, 11]).

We then apply this structure theorem to enumerate the labelled graphs in $\mathcal{F}$ according to the number of vertices and edges. Other results include a bound on the number of edges of graphs in $\mathcal{F}$, which is reminescent of planar graphs, and the enumeration of labelled homeomorphically irreducible graphs in $\mathcal{F}$.

In Section 2, we state and prove the structure theorem for the class $\mathcal{F}$. A general recursive decomposition for non-planar $K_{3,3}$-free graphs is described in Wagner [22] and Kelmans [14]. We recall the results of Fellows and Kaschube [7] and Gagarin and Kocay [9] on the structure of non-planar graphs containing a K_5-subdivision of a special type and cite from [9] the characterization of 2-connected non-planar $K_{3,3}$-free projective-planar graphs (the class $\mathcal{F}$) as graphs obtained from K_5 by substituting planar networks for edges. We then prove the uniqueness of this decomposition which establishes Theorem 1.

In Section 3, we first review some basic notions and terminology of labelled graphical enumeration. The reader should have some familiarity with exponential generating functions and their operations (addition, multiplication and composition). See, for example, Bergeron, Labelle and Leroux [2], Goulden and Jackson [13], Stanley [18], or Wilf [25]. We use mixed generating functions of the form

$$\mathcal{G}(x,y) = \sum_{n\geq 0}\sum_{m\geq 0} g_{n,m} y^m \frac{x^n}{n!}, \tag{2}$$

where $g_{n,m}$ is the number of graphs in a given class $\mathcal{G}$, with m edges and on a set of vertices V_n of size n. The main result here is that the effect on generating functions of the edge substitution operation is given by

$$(\mathcal{G} \uparrow \mathcal{N})(x, y) = \mathcal{G}(x, \mathcal{N}(x, y)). \quad (3)$$

Use is made of the enumeration of the class P of 2-connected planar graphs by Bender, Gao and Wormald [3] and Bodirsky, Gröpl, and Kang [4], based on previous work of Mullin and Schellenberg [16] on labelled enumeration of 3-connected planar graphs.

In Section 4, we study a special class of two-pole networks, the class $\mathcal{R}$ of *series-parallel* networks. Note that parallel edges are not permitted here. We also consider the species of series-parallel graphs denoted by $\mathcal{G}_{\mathrm{sp}}$ [6, 8, 23]. Our presentation follows a more structural and intuitive approach, where the emphasis is put on the structure classes or species. Indeed their recursive definitions can be translated into functional equations satisfied by the species themselves and these relations are then expressed in terms of their generating functions. Moreover, many computations can be carried out and understood at the species level, before taking the generating functions.

Attention is given, in Section 5, to the enumeration of the class $H_{\mathcal{F}}$ of homeomorphically irreducible graphs in $\mathcal{F}$. Here again the edge substitution operation $H \uparrow \mathcal{R}$ plays a central role, where $\mathcal{R}$ represents the class of series-parallel networks. We introduce a new general iterative scheme for the computation of the generating series $H(x, y)$ satisfying an identity of the form

$$B(x, y) = H(x, R(x, y)), \quad (4)$$

where $B(x, y)$ and $R(x, y)$ are known. This scheme is also applied to enumerate the class H_P of homeomorphically irreducible 2-connected planar graphs.

Finally, a short concluding section is devoted to some related questions. For example, the class $C_{\mathcal{F}}$ of connected projective-planar (non-planar) $K_{3,3}$-free graphs is studied and asymptotic questions are touched on. Extensions to the class $\mathcal{T}$ of non-planar toroidal $K_{3,3}$-free graphs and to unlabelled enumeration are also considered.

Numerical results appear in six tables giving the number of labelled graphs of the families $\mathcal{F}$, $H_{\mathcal{F}}$ and H_P, for example, for $n \leq 16$ and $m \leq 42$ for the class $H_{\mathcal{F}}$. Results for connected graphs in $\mathcal{F}$ are also given. The calculations were done with *Maple 9.5* software on Apple Macintosh computers.

2 A structure theorem for $K_{3,3}$-free projective-planar graphs

By convention, the graph K_2 is considered as a 2-connected (non-separable) graph in this paper. A *two-pole network* (or more simply, a *network*) is a connected graph N with two distinguished vertices 0 and 1, such that the graph $N \cup 01$ is 2-connected, where the notation $N \cup ab$ is used for the graph obtained from N by adding the edge ab if it is not already there. The vertices 0 and 1 are called the *poles* of N, and all the other vertices of N are called *internal* vertices.

We define an operator τ acting on 2-pole networks, $N \longmapsto \tau \cdot N$, which interchanges the poles 0 and 1. A class $\mathcal{N}$ of networks is called *symmetric* if $N \in \mathcal{N} \Longrightarrow \tau \cdot N \in \mathcal{N}$.

Definition. Let $\mathcal{G}$ be a class of graphs and $\mathcal{N}$ be a symmetric class of networks. We denote by $\mathcal{G} \uparrow \mathcal{N}$ the class of pairs of graphs (G, G_0), such that

1. the graph G_0 is in $\mathcal{G}$ (called the *core*),
2. the vertex set $V(G_0)$ is a subset of $V(G)$,
3. there exists a family $\{N_e : e \in E(G_0)\}$ of networks in $\mathcal{N}$ (called the *components*) such that the graph G can be obtained from G_0 by substituting N_e for each edge $e \in E(G_0)$, identifying the poles of N with the extremities of e according to some orientation.

An example of a $(\mathcal{G} \uparrow \mathcal{N})$-structure (G, G_0), with $\mathcal{G} = P_4$, the class of path-graphs of order 4, and $\mathcal{N}$ = the class of all networks, is given in Figure 2.

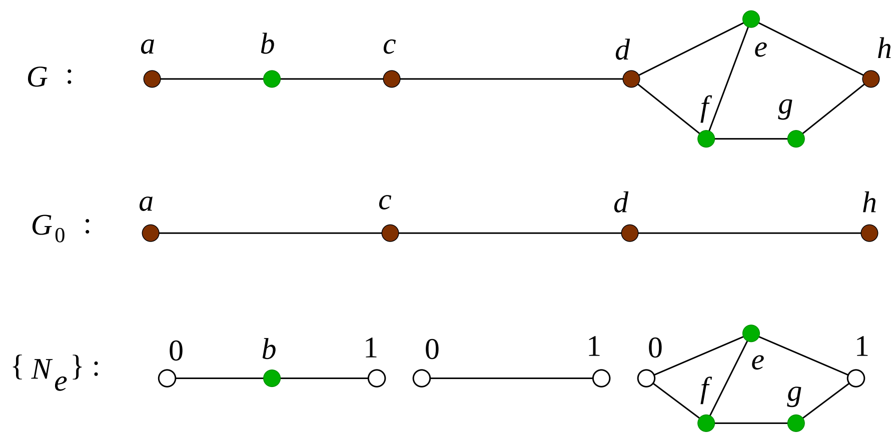

Figure 2: Example of a $(P_4 \uparrow \mathcal{N})$-structure (G, G_0)

The substitution of a network N_e for an edge e of G_0 is similar to the 2-sum operation defined for matroids and graphs by Seymour in [17]. One difference is that when the edge 01 is absent from the network N_e, the corresponding edge e is also absent from the resulting graph G.

We say that the composition $\mathcal{G} \uparrow \mathcal{N}$ is *canonical* if for any structure $(G, G_0) \in \mathcal{G} \uparrow \mathcal{N}$, the core $G_0 \in \mathcal{G}$ is uniquely determined by the graph G. In this case, we can identify $\mathcal{G} \uparrow \mathcal{N}$ with the class of resulting graphs G.

A network N is *strongly planar* if the graph $N \cup 01$ is planar. Denote by $\mathcal{N}_P$ the class of strongly planar networks. Let K_5 denote the class of complete graphs with 5 vertices.

Theorem 1 *The class $\mathcal{F}$ of 2-connected non-planar projective-planar $K_{3,3}$-free graphs can be expressed as a canonical composition*

$$\mathcal{F} = K_5 \uparrow \mathcal{N}_P. \tag{5}$$

Proof. We use the following previously established results. Following Diestel [5], a subgraph isomorphic to a K_5-subdivision is denoted by TK_5. Let G be a non-planar 2-connected graph with a TK_5. The vertices of degree 4 in TK_5 are called *corners* and the vertices of degree 2 are the *inner vertices* of TK_5. A path connecting two corners and containing no other corner is called a *side* of the K_5-subdivision. Note that two sides of the same TK_5 can have at most one common corner and no common inner vertices. A path p in G such that one endpoint is an inner vertex of TK_5, the other endpoint is on a different side of TK_5 and all other vertices and edges lie in $G \backslash TK_5$ is called a *shortcut* of the K_5-subdivision. A vertex $u \in G \backslash TK_5$ is called a 3-*corner vertex* with respect to TK_5 if $G \backslash TK_5$ contains internally disjoint paths from u to at least three corners of the K_5-subdivision.

The following proposition is proved in a different context:

Proposition 1 (Asano [1],[7, 9]) *Let G be a non-planar graph with a K_5-subdivision TK_5 for which there is either a shortcut or a 3-corner vertex. Then G contains a $K_{3,3}$-subdivision.*

Proposition 2 ([7, 9]) *Let G be a 2-connected graph with a TK_5 having neither a shortcut nor a 3-corner vertex. Let K denote the set of corners of TK_5. Then any connected component C of $G \backslash K$ contains inner vertices of at most one side of TK_5 and C is adjacent to exactly two corners of TK_5 in G.*

Given a graph G satisfying the hypothesis of Proposition 2, a *side component* of TK_5 is defined as a subgraph of G induced by a pair of corners a and b of TK_5 and all connected components of $G \backslash K$ which are adjacent to both a and b. Notice that side components or the entire graph G can contain a $K_{3,3}$-subdivision. For example, see Figure 3. However, if G has no $K_{3,3}$-subdivisions, then Proposition 2 can be applied in virtue of Proposition 1.

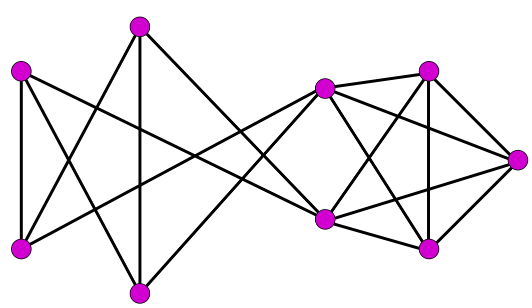

Figure 3: *A graph containing subdivisons of $K_{3,3}$ and K_5.*

Corollary 1 ([7, 9]) *For a 2-connected graph G with a TK_5 having no shortcut or 3-corner vertex, two side components of TK_5 in G have at most one vertex in common. The common vertex is the corner of intersection of two corresponding sides of TK_5.*

Thus we see that a graph G satisfying the hypothesis of Proposition 2 can be decomposed into side components corresponding to the sides of TK_5. Each side component H contains exactly two corners a and b corresponding to a side of TK_5. If the edge ab between the corners is not in H, we add it to H to obtain $H \cup ab$. Otherwise $H \cup ab = H$. We call $H \cup ab$ an *augmented side component* of TK_5.

If G is a 2-connected non-planar $K_{3,3}$-free graph, we can apply Proposition 2 and Corollary 1 to decompose G into side components of a TK_5. Notice that with 6 or more vertices, these graphs are not 3-connected since such a 3-connected non-planar graph contains a $K_{3,3}$-subdivision (see, for example, [1]).

Theorem 2 ([9]) *A 2-connected non-planar $K_{3,3}$-free graph G is projective-planar if and only if G contains a K_5-subdivision TK_5 such that each augmented side component of TK_5 is planar.*

The proof is based on properties of the two embeddings of the complete graph K_5 in the projective plane shown in Figure 1. Theorem 2 reduces projective-planarity testing to planarity testing if there is no $K_{3,3}$-subdivision in the graph. Theorem 2 can be strengthened to give the following equivalent form of Theorem 1. We say that a side component H is *strongly planar* if the augmented side component $H \cup ab$ is planar. This is coherent with the previously defined concept of strongly planar network.

Theorem 3 *A 2-connected non-planar projective-planar $K_{3,3}$-free graph G has a unique decomposition into strongly planar side components of a TK_5.*

Proof. By Proposition 2, the set of corners K of TK_5 completely defines a decomposition into the side components. Therefore it is sufficient to show that any other K_5-subdivision TK_5' in G shares the same set K of corners with TK_5. Since each augmented side component of TK_5 in G is planar, all corners of TK_5' cannot be contained in any particular side component. Suppose that a corner a of TK_5' is not in K. Then a is in a side component S of TK_5. Recall that there should be four disjoint paths from a to the four other corners of TK_5'. Since there is no shortcut or 3-corner vertex of TK_5 in G, the side component S of TK_5 must contain at least 2 other corners of TK_5', say b and c. Now consider a corner d of TK_5' that is not in the side component S of TK_5. Three of the sides adjacent to d must connect d to the three corners a, b and c of TK_5'. However the disjoint sides da, db and dc of TK_5' must share two corners of the side component S of TK_5, a contradiction. ∎

This concludes the proof of Theorem 1. ∎

A corollary to Euler's formula for the plane says that a planar graph with $n \geq 3$ vertices can have at most $3n - 6$ edges (see, for example, [5]). Let us state this for 2-connected planar graphs with n vertices and m edges as follows:

$$m \leq \begin{cases} 3n-5 & \text{if } n=2 \\ 3n-6 & \text{if } n \geq 3 \end{cases}. \tag{6}$$

In fact, $m = 3n - 5 = 1$ if $n = 2$. The generalized Euler Formula (see, for example, [19]) implies that a projective-planar graph G with n vertices can have up to $3n - 3$ edges. An arbitrary $K_{3,3}$-free graph G is known to have at most $3n-5$ edges (see [1]). However we show here that projective-planar $K_{3,3}$-free graphs satisfy the following stronger relation, which is similar to that of planar graphs.

Proposition 3 *The number m of edges of a non-planar projective-planar $K_{3,3}$-free n-vertex graph G satisfies $m = 3n-5 = 10$ if $n = 5$, and*

$$m \leq 3n-6 \text{ if } n \geq 6. \tag{7}$$

Proof. It is sufficient to prove the result for 2-connected graphs. By Theorem 2, each augmented side component S_i of G, $i = 1, 2, \ldots, 10$, satisfies the condition (6) with $n = n_i$, the number of vertices and $m = m_i$, the number of edges of S_i, $i = 1, 2, \ldots, 10$. Since each corner of TK_5 is in precisely 4 side components, we have $\sum_{i=1}^{10} n_i = n+15$ and we obtain, by summing these 10 inequalities,

$$m = \sum_{i=1}^{10} m_i \leq \begin{cases} 3\sum_{i=1}^{10} n_i - 50 = 3(n+15) - 50 = 3n-5 & \text{if } n = 5 \\ \\ 3\sum_{i=1}^{10} n_i - 51 = 3(n+15) - 51 = 3n-6 & \text{if } n \geq 6 \end{cases},$$

since $n = 5$ iff $n_i = 2$, $i = 1, 2, \ldots, 10$, and $n \geq 6$ if and only if at least one $n_j \geq 3$, $j = 1, 2, \ldots, 10$. ■

Notice that Corollary 8.3.5 of [5] implies that graphs without a K_5-subdivision also can have at most $3n-6$ edges.

3 Initial enumerative results

We now consider the labelled enumeration of projective-planar $K_{3,3}$-free graphs according to the numbers of vertices and edges. We first review some basic notions and terminology of labelled enumeration. The reader should have some familiarity with exponential generating functions and their operations (addition, multiplication and composition). See [2], [13], [18], or [25].

By a *labelled* graph, we mean a simple graph $G = (V, E)$ where the set of vertices $V = V(G)$ is itself the set of labels and the labelling function is the identity function. V is called the *underlying set* of G. An edge e of G then consists of an unordered pair $e = uv$ of elements of V and $E = E(G)$ denotes the set of edges of G. If W is another set and $\sigma : V \tilde{\rightarrow} W$ is a bijection, then any graph $G = (V, E)$ on V can be transformed into a graph $G' = \sigma(G) = (W, \sigma(E))$, where $\sigma(E) = \{\sigma(e) = \sigma(u)\sigma(v) \,|\, e \in E\}$. We say that G' is obtained from G by *vertex relabelling* and that σ is a graph *isomorphism* $G \tilde{\rightarrow} G'$. An *unlabelled graph* is then seen as an isomophism class γ of labelled graphs. We write $\gamma = \gamma(G)$ if γ is the isomorphism class of G. By the *number of ways to label* an unlabelled graph $\gamma(G)$, where $G = (V, E)$, we mean the number of distinct graphs G' on the underlying set V which are isomorphic to G. Recall that this number is given by $n!/|\mathrm{Aut}(G)|$, where $n = |V|$ and $\mathrm{Aut}(G)$ denotes the automorphism group of G.

A *species* of graphs is a class of labelled graphs which is closed under vertex relabellings. Thus any class $\mathcal{G}$ of unlabelled graphs gives rise to a species, also denoted by $\mathcal{G}$, by taking the set union of the isomorphism classes in $\mathcal{G}$. For any species $\mathcal{G}$ of graphs, we introduce its *mixed (exponential) generating function* $\mathcal{G}(x, y)$ as the formal power series

$$\mathcal{G}(x,y) = \sum_{n\geq 0} g_n(y)\frac{x^n}{n!}, \quad \text{with} \quad g_n(y) = \sum_{m\geq 0} g_{n,m}y^m, \tag{8}$$

where $g_{n,m}$ is the number of graphs in $\mathcal{G}$ with m edges over a given set of vertices V_n of size n. Here y is a formal variable which acts as an edge counter. For example, for the species $\mathcal{G} = K = \{K_n\}_{n\geq 0}$ of complete graphs, we have

$$K(x,y) = \sum_{n\geq 0} y^{\binom{n}{2}} x^n/n!, \tag{9}$$

while for the species $\mathcal{G} = \mathcal{G}_a$ of all simple graphs, we have $\mathcal{G}_a(x, y) = K(x, 1 + y)$. Another example is the class of *discrete* graphs (i.e. with no edges), which we denote by $\mathbb{E}$ (for French "Ensemble") since these are just sets of vertices, and we have $\mathbb{E}(x, y) = \sum_{n\geq 0} x^n/n! = \exp(x)$.

A species of graphs is *molecular* if it contains only one isomorphism class. Examples include the class K_1 of one-vertex graphs, which is denoted by X and satisfies $X(x, y) = x$, and the class K_2, with $K_2(x, y) = yx^2/2$. In general, for a molecular species $\gamma = \gamma(G)$, where G has n vertices and m edges, we have $\gamma(x,y) = \frac{y^m n!}{|\mathrm{Aut}(G)|}x^n/n! = y^m x^n/|\mathrm{Aut}(G)|$. For example, we have

$$K_5(x,y) = x^5y^{10}/5! \tag{10}$$

For two-pole networks, only the internal vertices form the underlying set for the purpose of enumeration and for species considerations. In particular, the mixed generating function of a class (or species) $\mathcal{N}$ of networks is defined by

$$\mathcal{N}(x,y) = \sum_{n\geq 0} \nu_n(y)\frac{x^n}{n!}, \quad \text{with} \quad \nu_n(y) = \sum_{m\geq 0} \nu_{n,m}y^m, \tag{11}$$

where $\nu_{n,m}$ is the number of networks in $\mathcal{N}$ with m edges with a given set of internal vertices V_n of size n.

There is an operator τ acting on two-pole networks, $N \longmapsto \tau \cdot N$, which interchanges the poles 0 and 1. A species $\mathcal{N}$ of networks is called *symmetric* if $N \in \mathcal{N} \Longrightarrow \tau \cdot N \in \mathcal{N}$. Examples of symmetric species of networks include the class $\mathcal{N}_P$ of strongly planar networks and the class $\mathcal{R}$ of series-parallel networks described in the next section.

Proposition 4 (T. Walsh [23]) *Let $\mathcal{G}$ be a species of graphs and $\mathcal{N}$ be a symmetric species of networks. Then we have*

$$(\mathcal{G} \uparrow \mathcal{N})(x,y) = \mathcal{G}(x, \mathcal{N}(x,y)). \tag{12}$$

Proof. If (G, G_0) is a $(\mathcal{G} \uparrow \mathcal{N})$-structure where the core graph G_0 has k edges, thus contributing a term y^k to $\mathcal{G}(x, y)$, we can assume that the underlying set of G_0 is linearly ordered. We say that the substitution of a network N for an edge $e = ab$, with $a < b$, is *coherent* if the pole 0 of N is identified with a and the pole 1, with b. Since the class $\mathcal{N}$ is symmetric, it is sufficient to restrict ourselves to coherent substitutions. Moreover, we can order the edges of G_0 lexicographically so that the process of edge subtitution is uniquely determined by a list of k disjoint networks in $\mathcal{N}$. Since these lists are counted by $\mathcal{N}^k(x, y)$, formula (12) follows. ■

Corollary 2 *The mixed generating function $\mathcal{F}(x, y)$ of labelled 2-connected non-planar projective-planar $K_{3,3}$-free graphs is given by*

$$\mathcal{F}(x, y) = \frac{x^5 \mathcal{N}_P^{10}(x, y)}{5!}. \tag{13}$$

Proof. This follows from Theorem 1, Proposition 4, and the fact that $K_5(x, y) = x^5 y^{10}/5!$. ■

There remains to compute the generating series $\mathcal{N}_P(x, y)$. If $\mathcal{M}$ is a class of networks which do not contain the edge 01, then we denote by $y\,\mathcal{M}$ the class obtained by adding this edge to all the networks of $\mathcal{M}$. Observe that there are two distinct networks on the empty set, namely the trivial network $\mathbb{1}$, consisting of two isolated poles 0 and 1, and the one edge network $y\mathbb{1}$.

Now let B be a given species of 2-connected graphs containing K_2, for example $B = B_a$, the class of all 2-connected graphs, $B = \{K_2\}$ or, more importantly here, $B = P$, the class of all 2-connected *planar* graphs. We denote by $B^{(y)}$ the species of graphs obtained by selecting and removing an edge in all possible ways from graphs in B . Note that

$$B^{(y)}(x, y) = \frac{\partial}{\partial y} B(x, y). \tag{14}$$

If, moreover, the endpoints of the selected edge are unlabelled and numbered 0 and 1, in all possible ways, the resulting class of networks is denoted by $B_{0,1}$. Relabelling the two poles yields the identity

$$x^2 B_{0,1}(x, y) = 2\, B^{(y)}(x, y). \tag{15}$$

Finally, we introduce the species of networks $\mathcal{N}_B$ associated to the class B by the formula

$$\mathcal{N}_B = B_{0,1} + y B_{0,1} - \mathbb{1} = (1 + y) B_{0,1} - \mathbb{1}. \tag{16}$$

Thus, the generating function of $\mathcal{N}_B$ is given by

$$\mathcal{N}_B(x, y) = (1 + y) \frac{2}{x^2} \frac{\partial}{\partial y} B(x, y) - 1. \tag{17}$$

Let P denote the species of 2-connected planar graphs. Then the associated class $\mathcal{N}_P$ of networks described above is precisely the class of strongly planar networks. Methods

n	m	$f_{n,m}$	n	m	$f_{n,m}$	n	m	$f_{n,m}$
5	10	1	11	16	1664863200	13	18	1261490630400
6	11	60	11	17	17556739200	13	19	21330659750400
6	12	60	11	18	78956539200	13	20	159781461840000
7	12	2310	11	19	202084621200	13	21	713882464495200
7	13	5250	11	20	334016949420	13	22	2168012582255520
7	14	3150	11	21	387489624060	13	23	4841896937557680
7	15	210	11	22	335202677040	13	24	8367745313108610
8	13	73920	11	23	221055080400	13	25	11501380415300490
8	14	283920	11	24	107529691500	13	26	12648862825333020
8	15	380240	11	25	35726852700	13	27	11024998506341820
8	16	205520	11	26	7205814000	13	28	7476620617155690
8	17	40320	11	27	663616800	13	29	3846042558007650
8	18	5040	12	17	45664819200	13	30	1446703666808400
9	14	2162160	12	18	617512896000	13	31	374735495534400
9	15	12383280	12	19	3642195110400	13	32	59680805184000
9	16	27592740	12	20	12576897194400	13	33	4401725328000
9	17	30616740	12	21	28943910959040	14	19	35321737651200
9	18	18419940	12	22	48122268218640	14	20	732123289497600
9	19	6656580	12	23	61023477279600	14	21	6797952466905600
9	20	1678320	12	24	60601323301200	14	22	38137563765100800
9	21	196560	12	25	46937904829200	14	23	147357768378300480
10	15	60540480	12	26	27584940398400	14	24	423597368531216880
10	16	481572000	12	27	11793019392000	14	25	951297908961680280
10	17	1578301200	12	28	3448102996800	14	26	1715806516686001740
10	18	2810039400	12	29	615367368000	14	27	2511869870973763300
10	19	3055603320	12	30	50494752000	14	28	2981167142609535880
10	20	2214739800				14	29	2845977828319866240
10	21	1155735000				14	30	2159624129854611420
10	22	432356400				14	31	1281613625914642020
10	23	98809200				14	32	580974136160418000
10	24	10281600				14	33	194019911828542800
						14	34	44947147269024000
						14	35	6446992644892800
						14	36	431053060838400

Table 1: The number $f_{n,m}$ of labelled non-planar projective-planar 2-connected graphs without a $K_{3,3}$-subdivision (having n vertices and m edges).

for computing the generating function $P(x,y)$ of labelled 2-connected planar graphs are described in [3] and [4]. Both methods are based on the network decomposition of [20] which is also stated for planar graph embeddings in [21]. The decomposition allows to count the 2-connected planar graphs via labelled 3-connected planar graphs whose counting can be derived from [16].

Formulas (17) and (13) can then be used to compute $\mathcal{N}_P(x,y)$ and $\mathcal{F}(x,y)$. Numerical results are presented in Tables 1 and 2,where $\mathcal{F}(x,y) = \sum_{n\geq 5}\sum_m f_{n,m}x^n y^m/n!$ and $f_n = \sum_m f_{n,m}$.

4 Series-parallel networks and graphs

In this section, we study a special class of two-pole networks, the class $\mathcal{R}$ of *series-parallel* networks (also called two-terminal directed series-parallel networks). Note that parallel edges are not permitted here. We also consider the species of series-parallel graphs, denoted by $\mathcal{G}_{\text{sp}}$. See, for example, [6], [8] and [23].

n	f_n
5	1
6	120
7	10920
8	988960
9	99706320
10	11897978400
11	1729153068720
12	306003079514880
13	64657337524631280
14	15890834362452489440
15	4435396700216405763840
16	1379778057502074926142720
17	471689356958791639787042560
18	175335742043846629500183667200
19	70291642269058321415718042668160
20	30195035473057938652243866755197440

Table 2: The number f_n of labelled non-planar projective-planar 2-connected graphs without a $K_{3,3}$-subdivision (having n vertices).

It is assumed that the poles of a network N are distinct from those of any other network. There are two main operations on two-pole networks: parallel composition and series composition. Let S be a finite set of disjoint networks which are not equal to $\mathbb{1}$ and do not contain the edge 01. The *parallel composition* of S is the network obtained by taking the union of the graphs in S where, moreover, all the 0-poles are fusioned into one 0-pole and similarly for the 1-poles. By convention, the parallel composition of an empty set of networks is the trivial network $\mathbb{1}$. If $\mathcal{N}$ is a species of networks which are distinct from $\mathbb{1}$ and have non-adjacent poles and if each network in a class $\mathcal{M}$ can be viewed unambiguously as a parallel composition of networks in $\mathcal{N}$, then the result can be expressed as a species composition $\mathcal{M} = \mathbb{E}(\mathcal{N})$, and we have

$$\mathcal{M}(x, y) = \exp(\mathcal{N}(x, y)). \tag{18}$$

Note that the class $\mathcal{N}$ is then included in $\mathcal{M}$ and that $\mathbb{1}$ is in $\mathcal{M}$.

Let M and N be two non-trivial disjoint networks. The *series composition* $M \cdot_s N$ of M followed by N is a network whose underlying set is the union of the underlying sets of M and N plus an extra element. It is obtained by taking the graph union of M and N where moreover the 1-pole of M is fusioned with the 0-pole of N and this *connecting vertex* is labelled by the extra element. The *series composition* $\mathcal{M} \cdot_s \mathcal{N}$ of two species of networks $\mathcal{M}$ and $\mathcal{N}$ not containing $\mathbb{1}$ is the class obtained by taking all series compositions $M \cdot_s N$ with $M \in \mathcal{M}$ and $N \in \mathcal{N}$. If moreover the two components $M \in \mathcal{M}$ and $N \in \mathcal{N}$ are uniquely determined by the resulting network $M \cdot_s N$, the species $\mathcal{M} \cdot_s \mathcal{N}$ can be expressed as the species product $\mathcal{M}X\mathcal{N}$, where the factor X corresponds to the connecting vertex, and we have

$$(\mathcal{M} \cdot_s \mathcal{N})(x, y) = x\mathcal{M}(x, y)\mathcal{N}(x, y). \tag{19}$$

The species $\mathcal{R}$ of *series-parallel* networks can be defined recursively as the smallest class of networks containing the one-edge network $y\mathbb{1}$ and closed under series and parallel compositions. We partition $\mathcal{R}$ as $\mathcal{R} = \mathcal{S} + \mathcal{P}$, where $\mathcal{S}$ represents the species of *essentially series* networks (i.e. of the form (20) below) and $\mathcal{P} = \mathcal{R} - \mathcal{S}$ is the complementary class, of *essentially parallel* networks. These classes are characterized recursively by the following functional equations involving series and parallel composition:

$$\mathcal{S} = \mathcal{P} \cdot_s \mathcal{R} = \mathcal{P}X\mathcal{R}. \tag{20}$$

$$\mathcal{R} = (1+y)\mathbb{E}(\mathcal{S}) - \mathbb{1}. \tag{21}$$

From these two equations, we deduce

$$\mathcal{R} = \mathcal{S} + \mathcal{P} = \mathcal{P}X\mathcal{R} + \mathcal{P} = \mathcal{P}(1 + X\mathcal{R}) \;\Rightarrow\; \mathcal{P} = \mathcal{R}/(1 + X\mathcal{R}) \tag{22}$$

and

$$\mathcal{R} = (1+y)\mathbb{E}(\mathcal{P}X\mathcal{R}) - \mathbb{1} = (1+y)\mathbb{E}(\frac{X\mathcal{R}^2}{1+X\mathcal{R}}) - \mathbb{1} \tag{23}$$

and for the generating functions,

$$\mathcal{R}(x,y) = (1+y)\exp\left(\frac{x\mathcal{R}^2(x,y)}{1+x\mathcal{R}(x,y)}\right) - 1. \tag{24}$$

The series $\mathcal{R}(x,y)$ can be computed recursively using (24).

Now a *series-parallel graph* is a 2-connected graph which is either an edge or can be obtained from a series-parallel network by adding the edge 01 (if not already present) and labelling the poles. Series-parallel graphs can be characterized as 2-connected graphs without a K_4-subdivision (see [6]). The class of series-parallel graphs is denoted by $\mathcal{G}_{\text{sp}}$. It is easy to see that the networks induced by series-parallel graphs are precisely the series-parallel networks, i.e. that

$$\mathcal{N}_{\mathcal{G}_{\text{sp}}} = \mathcal{R}. \tag{25}$$

This implies, using (17), that

$$\mathcal{G}_{\text{sp}}(x,y) = \frac{x^2}{2}\int_0^y \frac{\mathcal{R}(x,t)+1}{1+t}\,dt. \tag{26}$$

5 Homeomorphically irreducible graphs

A graph is called *homeomorphically irreducible* if it contains no vertex of degree 2. For a graph G embeddable in a surface, any subdivision of G is trivially embeddable in the same surface. Therefore, it is interesting to count graphs embeddable in a surface that are minimal with respect to the operation of subdivision, i.e. homeomorphically irreducible graphs. Here we do this for the classes P of 2-connected planar graphs and $\mathcal{F}$ of 2-connected non-planar projective-planar $K_{3,3}$-free graphs, applying the method of Walsh ([23]) as follows.

Any 2-connected graph G is either a series-parallel graph or contains a unique 2-connected homeomorphically irreducible core $C(G)$, which is different from K_2, and unique components $\{N_e\}_{e\in E(C(G))}$ which are series-parallel networks, whose composition gives G. Let $\mathcal{B}$ be a species of 2-connected graphs. Denote by $H_{\mathcal{B}}$ the class of graphs which are homeomorphically irreducible cores of graphs in $\mathcal{B}$. Also set $\mathcal{B}_{\mathrm{sp}} = \mathcal{B} \cap \mathcal{G}_{\mathrm{sp}}$ which is the class of series-parallel graphs in $\mathcal{B}$.

Proposition 5 *Let $\mathcal{B}$ be a species of 2-connected graphs such that*

1. *$H_{\mathcal{B}}$ is contained in $\mathcal{B}$,*
2. *$\mathcal{B}$ is closed under edge substitution by series-parallel networks, i.e. $\mathcal{B} \uparrow \mathcal{R}$ is contained in $\mathcal{B}$.*

Then we have

$$\mathcal{B} = \mathcal{B}_{\mathrm{sp}} + H_{\mathcal{B}} \uparrow \mathcal{R}, \tag{27}$$

and the composition $H_{\mathcal{B}} \uparrow \mathcal{R}$ is canonical.

For the generating functions, it follows that

$$\mathcal{B}(x,y) = \mathcal{B}_{\mathrm{sp}}(x,y) + H_{\mathcal{B}}(x, \mathcal{R}(x,y)), \tag{28}$$

from which the series $H_{\mathcal{B}}(x,y)$ can be computed, in virtue of the following lemma:

Lemma 1 *Let $B(x,y)$ and $R(x,y)$ be two-variable formal power series such that $R(x,y) = y + O(y^2)$. Then there exists a unique formal power series $H(x,y)$ such that*

$$B(x,y) = H(x, R(x,y)). \tag{29}$$

Moreover, $H(x,y)$ can be expressed as

$$H(x,y) = \sum_{i\geq 0} (-1)^i \Delta_R^i B(x,y), \tag{30}$$

where Δ_R is an operator defined on two-variable formal power series $F(x,y)$ by

$$\Delta_R F(x,y) = F(x, R(x,y)) - F(x,y).$$

Proof. The first statement follows from the fact that under the hypothesis, the series $R(x,y)$, viewed as a formal power series in the variable y, is invertible under composition. The equation (30) then follows from the observation that equation (29) is equivalent to $B(x,y) = (I + \Delta_R)H(x,y)$, where I denotes the identity operator. Details are left to the reader. ∎

Walsh uses Proposition 5 to enumerate all labelled homeomorphically irreducible 2-connected graphs in [23]. Here we give two other applications. First, we take $\mathcal{B} = P$, the class of 2-connected planar graphs. In this case, $H_{\mathcal{B}} = H_P$ is the class of 2-connected

n	m	$H_P(n,m)$	n	m	$H_P(n,m)$	n	m	$H_P(n,m)$
2	1	1	10	15	5700240	13	20	1845922478400
4	6	1	10	16	297561600	13	21	74599125400800
5	8	15	10	17	2930596200	13	22	989130437895600
5	9	10	10	18	12343659300	13	23	6630351423696000
6	9	60	10	19	28301918400	13	24	26817549328369800
6	10	477	10	20	38982967065	13	25	71682957811565100
6	11	585	10	21	33331061925	13	26	133187371098982200
6	12	195	10	22	17392158000	13	27	176696593868094300
7	11	4410	10	23	5088258000	13	28	169059691482031350
7	12	23520	10	24	641277000	13	29	116043129855402750
7	13	37800	11	17	1659042000	13	30	55840786515914400
7	14	24570	11	18	41399542800	13	31	17912090131135200
7	15	5712	11	19	340745605200	13	32	3443842956153600
8	12	13440	11	20	1407085287300	13	33	300495408595200
8	13	332640	11	21	3435723903150	14	21	4217639025600
8	14	1543860	11	22	5355953687700	14	22	594554129769600
8	15	2917740	11	23	5504170275450	14	23	15443454480854400
8	16	2708160	11	24	3728003340600	14	24	172534400373535200
8	17	1237320	11	25	1606084131960	14	25	1084459459555672200
8	18	223440	11	26	399801679200	14	26	4361354314691635800
9	14	2177280	11	27	43859692800	14	27	12056896085921586900
9	15	28962360	12	18	3996669600	14	28	23900258922899609250
9	16	126168840	12	19	362978431200	14	29	34803857129521580100
9	17	266535360	12	20	6150939628680	14	30	37649908340175226095
9	18	311551380	12	21	44916513919200	14	31	30254152933093434345
9	19	207170460	12	22	183180611357100	14	32	17843305708519691400
9	20	73710000	12	23	470167225050600	14	33	7512030324951352200
9	21	10929600	12	24	807689258734050	14	34	2139154225643635200
			12	25	956591057815470	14	35	369529809669820800
			12	26	786477564207720	14	36	29262949937020800
			12	27	442142453075400			
			12	28	162493688649600			
			12	29	35240506963200			
			12	30	3424685806080			

Table 3: The number $H_P(n,m)$ of labelled 2-connected homeomorphically irreducible planar graphs (having n vertices and m edges).

planar graphs with no vertices of degree less than 3 and $H_P + K_2$ is the class of 2-connected homeomorphically irreducible planar graphs. Series-parallel graphs are known to be planar: they do not contain a subdivision of K_4, but K_5 and $K_{3,3}$ do. It follows that $P_S = P \cap \mathcal{G}_{\mathrm{sp}} = \mathcal{G}_{\mathrm{sp}}$. It is clear that the hypotheses of Proposition 5 are satisfied and we deduce from (27) that

$$P = \mathcal{G}_{\mathrm{sp}} + H_P \uparrow \mathcal{R} \tag{31}$$

where the composition is canonical. Taking the generating functions, we obtain:

Proposition 6 *The mixed generating functions of the species P of planar 2-connected graphs and H_P of homeomorphically irreducible graphs in P are related by the equation*

$$P(x,y) = \mathcal{G}_{\mathrm{sp}}(x,y) + H_P(x, \mathcal{R}(x,y)). \tag{32}$$

We have used this equation to compute the first terms of the series $H_P(x,y) = \sum_{n\geq 4}\sum_m H_P(n,m)x^n y^m/n!$ using Lemma 1. The results are presented in Tables 3 and 4, where $H_P(n) = \sum_m H_P(n,m)$. Notice that the computational results of Table 3 in

n	$H_P(n)$
4	1
5	25
6	1317
7	96012
8	8976600
9	1027205280
10	139315157730
11	21864486188160
12	3898841480307900
13	778680435365714700
14	172192746831203449890
15	41765231538761743574100
16	11024455369912310561835600
17	3146065407516184280981053200
18	965135197612755256313598822450
19	316731891055609655106993297185400
20	110718818921232836033343337842628500

Table 4: The number $H_P(n)$ of labelled 2-connected homeomorphically irreducible planar graphs (having n vertices).

comparison to those of [3] verify that any maximal planar graph with $n \geq 4$ vertices (and $3n-6$ edges) have all vertex degrees at least 3. In other words, for $n \geq 4$ and $m = 3n-6$, we have $H_P(n,m) = P(n,m)$.

Now we take $\mathcal{B} = \mathcal{F}$, the species of non-planar projective-planar 2-connected graphs without a $K_{3,3}$-subdivision. Then $H_{\mathcal{F}}$ is the class of homeomorphically irreducible graphs in $\mathcal{F}$ and $\mathcal{F}_{\text{sp}} = \mathcal{F} \cap \mathcal{G}_{\text{sp}}$ is empty. It is clear that the hypotheses of Proposition 5 are satisfied and we deduce from (27) that

$$\mathcal{F} = H_{\mathcal{F}} \uparrow \mathcal{R} \tag{33}$$

where the composition is canonical. Taking the generating functions, we obtain:

Proposition 7 *The generating functions $\mathcal{F}(x,y)$ and $H_{\mathcal{F}}(x,y)$ of labelled non-planar projective-planar 2-connected $K_{3,3}$-free graphs and those with no vertices of degree less than* 3 *(resp.) are related by the equation*

$$\mathcal{F}(x,y) = H_{\mathcal{F}}(x, \mathcal{R}(x,y)). \tag{34}$$

We have used this equation to compute the first terms of the series $H_{\mathcal{F}}(x,y) = \sum_{n\geq 5}\sum_m h_{n,m}x^n y^m/n!$. The results are presented in Tables 5 and 6, where $h_n = \sum_m h_{n,m}$. Notice that numbers in Table 6 are much smaller than corresponding numbers in Table 2. However, for $n \geq 7$ and $m = 3n-6$, the corresponding numbers in Tables 1 and 5 are the same. This verifies that maximal non-planar projective-planar $K_{3,3}$-free graphs have vertex degrees at least 3. That can be seen as a corollary to Theorem 2, Proposition 4 and the corresponding statement for maximal planar graphs.

There is an alternate way to compute the series $H_{\mathcal{F}}(x,y)$ which reduces computations significantly. The idea is to determine what are the side components that should be substituted into the edges of K_5 in order to obtain homeomorphically irreducible graphs

n	m	$h_{n,m}$	n	m	$h_{n,m}$	n	m	$h_{n,m}$
5	10	1	12	21	2025777600	15	25	7205830632000
7	14	210	12	22	44347564800	15	26	923081887728000
7	15	210	12	23	321609657600	15	27	21992072494392000
8	15	3360	12	24	1163155593600	15	28	226159100164998000
8	16	13440	12	25	2450459088000	15	29	1307760868616202000
8	17	15120	12	26	3214825059600	15	30	4819747766658224400
8	18	5040	12	27	2674115413200	15	31	12125014783013632500
9	16	15120	12	28	1375742491200	15	32	21647164674205570500
9	17	257040	12	29	400355524800	15	33	27986100453182371500
9	18	948780	12	30	50494752000	15	34	26352994794020744850
9	19	1372140	13	22	5772967200	15	35	17930879200055571750
9	20	861840	13	23	462940077600	15	36	8598757529558196000
9	21	196560	13	24	7019020008000	15	37	2759781730918212000
10	18	2116800	13	25	45946008108000	15	38	532536328868544000
10	19	23511600	13	26	167127038278200	15	39	46746961252992000
10	20	85428000	13	27	378396335116800	16	27	5038469339904000
10	21	145681200	13	28	563392705525650	16	28	277876008393984000
10	22	128898000	13	29	563307338043450	16	29	5018906168115840000
10	23	57531600	13	30	375765580990800	16	30	45917601694892928000
10	24	10281600	13	31	160783623000000	16	31	254980573605117360000
11	19	6652800	13	32	39987851904000	16	32	945355953679641504000
11	20	301039200	13	33	4401725328000	16	33	2474074000472282208000
11	21	2559249000	14	24	2746116172800	16	34	4725628918340842512000
11	22	9235749600	14	25	100222343020800	16	35	6713043612043772203200
11	23	17753412600	14	26	1207921516401600	16	36	7147503225193821890400
11	24	19736016300	14	27	7362246043152000	16	37	5690265588000873079200
11	25	12781345500	14	28	26958084641888400	16	38	3341472585422235264000
11	26	4491471600	14	29	64675702745854200	16	39	1406015363067771456000
11	27	663616800	14	30	106440372932493600	16	40	401357157281144064000
			14	31	122731243349715000	16	41	69663390944539392000
			14	32	99327282369915600	16	42	5553155150839296000
			14	33	55396720246467600			
			14	34	20312069633856000			
			14	35	4413395543356800			
			14	36	431053060838400			

Table 5: The number $h_{n,m}$ of labelled non-planar projective-planar 2-connected $K_{3,3}$-free homeomorphically irreducible graphs (having n vertices and m edges).

in $\mathcal{F}$. The question is to determine the class of networks $\mathcal{N}$ for which $H_{\mathcal{F}} = K_5 \uparrow \mathcal{N}$. A first try is to take the class $\mathcal{N} = \mathcal{N}_{H_P}$ of networks N such that $N \cup 01$ is a planar 2-connected graph with no vertices of degree less than 3. Following (17), we have

$$\mathcal{N}_{H_P}(x, y) = (1 + y)\frac{2}{x^2}\frac{\partial}{\partial y}H_P(x, y) - 1, \tag{35}$$

where $H_P(x, y)$ is given by Proposition 6. However, the degree requirements can be relaxed for the two poles of the network N since they will become identified with the corners of K_5 and will have degree at least 4. In particular, the one-edge network $y\mathbb{1}$ can be used as a side component. Another way to have a pole of degree one is to start with a network N of $\mathcal{N} = \mathcal{N}_{H_P}$ and *add a leg* at the 0-pole, the 1-pole, or both. This means taking the series compositions $y\mathbb{1} \cdot_s N$, $N \cdot_s y\mathbb{1}$, or $y\mathbb{1} \cdot_s N \cdot_s y\mathbb{1}$. In this way, we obtain the class $y\mathbb{1}X\mathcal{N}_{H_P} + \mathcal{N}_{H_P}Xy\mathbb{1} + y\mathbb{1}X\mathcal{N}_{H_P}Xy\mathbb{1} = (2y\mathbb{1}X + (y\mathbb{1}X)^2)\mathcal{N}_{H_P}$. Let us denote by Leg this operator of adding legs, with $\mathrm{Leg}(x, y) = 2yx + y^2x^2$. Now it is possible to join the

n	h_n
5	1
6	0
7	420
8	36960
9	3651480
10	453448800
11	67528553400
12	11697130922400
13	2306595939347700
14	509359060543132800
15	124356566550728011500
16	33226132945543622884800
17	9635384706205021006042800
18	3012126613564117021370798400
19	1009263543337906919842715535600
20	360698621436519186180018210552000

Table 6: The number of labelled non-planar projective-planar 2-connected $K_{3,3}$-free homeomorphically irreducible graphs (having n vertices).

poles by an edge to obtain one or two poles of degree 2, giving rise to the operator $y\mathrm{Leg}$, and to iterate this process. Hence we set

$$\mathcal{N}_\ell = (1+\mathrm{Leg})\left(\sum_{k\geq 0}(y\mathrm{Leg})^k\right)\mathcal{N}_{H_P}. \tag{36}$$

The generating series of this class of networks is given by

$$\mathcal{N}_\ell(x,y) = \frac{1+\mathrm{Leg}(x,y)}{1-y\mathrm{Leg}(x,y)}\mathcal{N}_{H_P}(x,y), \tag{37}$$

where $\mathcal{N}_{H_P}(x,y)$ is defined by (35), and we have the following proposition.

Proposition 8 *Let $\mathcal{N}_\ell$ be the species of networks defined by equation* (36)*. Then the species $H_\mathcal{F}$ of homeomorphically irreducible non-planar projective-planar 2-connected $K_{3,3}$-free graphs can be expressed as*

$$H_\mathcal{F} = K_5 \uparrow (\mathcal{N}_\ell + y\mathbb{1}) \tag{38}$$

and its generating series satisfies

$$H_\mathcal{F}(x,y) = \frac{x^5(\mathcal{N}_\ell(x,y)+y)^{10}}{5!}, \tag{39}$$

where $\mathcal{N}_\ell(x,y)$ is given by (37)*.*

6 Concluding Remarks

We have obtained the labelled enumeration of the class $\mathcal{F}$ of 2-connected non-planar projective-planar $K_{3,3}$-free graphs. In this short section, we mention some extensions which can be obtained of these results.

6.1 Connected graphs in $\mathcal{F}$

It is easy to deduce the labelled enumeration for the class $C_{\mathcal{F}}$ of 1-connected (i.e. connected) non-planar projective-planar $K_{3,3}$-free graphs. Indeed it suffices to attach arbitrary vertex-rooted connected planar graphs at each vertex of graphs in $\mathcal{F}$ in order to obtain all graphs in $C_{\mathcal{F}}$. More precisely, we have

$$C_{\mathcal{F}}(x, y) = \mathcal{F}(C_P^{\bullet}(x, y), y), \tag{40}$$

where $C_P^{\bullet}$ denotes the class of vertex rooted connected planar graphs.

Recall that P denotes the class of 2-connected planar graphs. Then it is well known (see [2], [12]) that

$$C_P^{\bullet}(x, y) = x \exp(P'(C_P^{\bullet}(x, y), y)), \tag{41}$$

where $P'(x, y) = \frac{\partial}{\partial x} P(x, y)$, from which $C_P^{\bullet}(x, y)$ and then $C_{\mathcal{F}}(x, y)$ can be computed. For example, setting $y = 1$, we obtain the following first numbers $|C_{\mathcal{F}}[n]|$ of labelled connected non-planar projective-planar $K_{3,3}$-free graphs with n vertices, $n = 5, 6, \ldots, 20$:

1, 150, 16800, 1809360, 206725050, 26484163020, 3942600552660, 694822388340960,
145100505844928205, 35439528882292735200,
9927470411345581984890, 3128005716477250367216640,
1090689073286188397027568380, 415560636438834909293721364320,
171338083303545263513720985887520, 75873636257232699557453120820157440.

6.2 Asymptotics

Using results of Bender, Gao and Wormald [3] and Giménez and Noy [12], it is easy to see that labelled non-planar $K_{3,3}$-free projective-planar 2-connected and 1-connected graphs share similar asymptotic behaviours, in particular, the same growth constants, as their planar counterparts.

6.3 Other extensions

Using methods of the present paper, we have been able to give a similar characterization for the class $\mathcal{T}$ of non-planar $K_{3,3}$-free toroidal 2-connected graphs. In this case, more complex *toroidal* cores can occur, whose class is denoted by $\mathcal{T}_C$, for which the equation $\mathcal{T} = \mathcal{T}_C \uparrow \mathcal{N}_P$ holds. See [10] for details.

Walsh in [24] has shown how to enumerate *unlabelled* graphs in a class which admits an unambiguous representation of the form $\mathcal{G} \uparrow \mathcal{N}$. Therefore the characterization of Theorem 1 also leads to the unlabelled enumeration of $K_{3,3}$-free projective-planar and toroidal graphs. This has been carried out in the paper [11].

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